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Second Order Random Differential Inclusion

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Abstract

In this paper, two existence theorems for the second order ordinary random differential inclusions are proved for convex case of random differential inclusions.

Keywords; Second order random differential inclusion, existence theorem, random solution etc.

Introduction

The topic of random differential inclusions is an important branch of multi-valued analysis and deals with the ordinary and partial random differential inclusions. In the present paper, I have to restrict ordinary differential inclusions only. The general expression for ordinary random differential inclusion is

$$\mathrm{L}(\omega)x \in \mathrm{N}\,(\omega)x$$

satisfying initial condition

$$x \in \mathrm{J}\ ,$$

or boundary conditions

$$x \in \mathrm{B}\,,$$

where $\mathrm{L}(\omega)$ is a linear differential operator defined by

$$\mathrm{L}(\omega)x = c_0(\omega)\frac{d^n}{dt^n}x(t,\omega) + ... + \frac{dx(t,\omega)}{dt} + c_n x(t,\omega)$$

where $c_i, i = 0,1,.....,n$ being the real measurable functions and $\mathrm{N}\,(\omega)$ is a Nemytsky operator defined on a suitable function spaces, and $\mathrm{N}\,(\omega)$ is given by

$$\mathrm{N}\,(\omega)x = F\left(t, x(t,\omega), \frac{dx(t,\omega)}{dt}, ... \frac{d^{n-1}}{dt^{n-1}}x(t,\omega), \omega\right).$$

Many of the classical existence results of ordinary differential inclusions can be carried over to random differential inclusions successfully under suitable modifications. The main difficulty in all above

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

existence results lies in the proof of measurability of the classical solution in stochastic variable $\omega \in \Omega$. The initial value problem (in short IVP) of ordinary first order random differential inclusion

$$\left.\begin{aligned} x'(t,\omega) &\in F(t, x(t,\omega), \omega) \\ x(0,\omega) &= q(\omega) \end{aligned}\right\}.$$

has been discussed in Papageorgiou for existence results under different conditions. In the present paper,I deal with different types of ordinary random differential inclusions for existence and existence of extremalsolutions under suitable conditions.

1. Statement of the Problem

Let $(\Omega, \mathrm{A}, \mu)$ be a complete σ-finite measure space and let *R(*= $\mathbb{R}$) be the real line. Let *P(* $\mathbb{R}$) denote the class of all non-empty subsets of $\mathbb{R}$ with property p. Given a closed and bounded interval $J = [0, T]$ and given two measurable functions $q_0, q_1 : \Omega \to \mathbb{R}$, consider the second order

$$\left.\begin{aligned} x''(t,\omega) &\in F\big(t, x(t,\omega), \omega\big) \quad a.e. \quad t \in J \\ x(0,\omega) &= q_0(\omega), x'(0,\omega) = q_1(\omega) \end{aligned}\right\} \tag{1}$$

for all $\omega \in \Omega$, where $F : J \times \mathbb{R} \times \Omega \to \mathrm{P}_p(\mathbb{R})$.

By a random solution of the RDI (3.1.1) on $J \times \Omega$ we mean a measurable function $x : \Omega \to AC^1(J, \mathbb{R})$ satisfying for each $\omega \in \Omega$, $x''(t,\omega) = v(t,\omega)$ for some measurable $v : \Omega \to L^1(J, \mathbb{R})$ such that $v(t,\omega) \in F\big(t, x(t,\omega), \omega\big)$ a.e. $t \in J$, where $AC^1(J, \mathbb{R})$ is the space of continuous real-valued functions whose first derivative is absolutely continuous on *J*.

The existence theorems for radon differential inclusions are generally proved using the topological fixed-point theory under certain compactness and measurability conditions of multi-valued functions on the right hand side of differential inclusions in question. A "priori bound method" is proved to be very much useful for proving the existence theorems for initial value problems of random differential inclusions. See for example, Deimling 'Hu and Papageorgiou and Dhage etc. When the right hand side multi-valued function is not convex-valued, the geometrical or algebraic multi-valued fixed-point theory is used for proving the existence theorem under certain Lipschitz and monotonicity conditions of multi-valued functions. In the present work, we will prove two existence results for convex and nonconvex case of second order random differential inclusions. Below in the following section, we give some preliminary definitions and some fundamental results that will be used in the subsequent part of this paper.

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

2. Auxiliary Results

Let $M(J,\mathbb{R})$ denote the class of real-valued measurable functions on *J* and let $C(J,\mathbb{R})$ denote the space of continuous real-valued functions on *J*. Let $L^1(J,\mathbb{R})$ denote the Banach space of Lebesgue integrable functions on *J* with norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1}=\int_0^T x(t)dt .$$

Let $F:J\times\mathbb{R}\times\mathbb{R}\times\Omega\to \mathrm{P}_p(\mathbb{R})$ be a multi-valued mapping. Then for only measurable function $x:\Omega\to C(J,\mathbb{R})$, let

$$S_F(\omega)(x)=\left\{v\in \mathrm{M}\left(\Omega, M(J,\mathbb{R})\right) \mid v(t,\omega)\in F\left(t,x(t,\omega),\omega\right)\ \ a.e.\ \ t\in J\right\}. \quad (2.1)$$

and

$$S_F^1(\omega)(x)=\left\{v\in \mathrm{M}\left(\Omega, L^1(J,\mathbb{R})\right) \mid v(t,\omega)\in F\left(t,x(t,\omega),\omega\right)\ \ a.e.\ \ t\in J\right\}. \quad (2.2.2)$$

This is our set of selection functions for *F* on J × $\mathbb{R}$ × Ω. The integral of the random multi-valued function *F* is defined as

$$\int_0^t F\left(s,x(s,\omega),\omega\right)ds=\left\{\int_0^t v(s,\omega)ds : v\in S_F^1(\omega)(x)\right\}.$$

Furthermore, if the integral $\int_0^t F\left(s,x(s,\omega),\omega\right)ds$ exists for every measurable function $x:\Omega\to C(J,\mathbb{R})$, then we say the multi-valued mapping *F* is Lebesgue integrable on *J*. We need the following definitions in the sequel.

<u>Definition</u> 2.1 A multi-valued mapping $\beta:\Omega\to \mathrm{P}_{cp}(\mathbb{R})$ is said to be measurable if for any $y\in\mathbb{R}$, the function $\omega\to d\left(y,F(\omega)\right)=\ \inf\left\{\left|y-x\right| : x\in F(\omega)\right\}$ is measurable.

Definition 2.2 A multi-valued mapping $F : J \times \mathbb{R} \times \Omega \to \mathrm{P}_{cp}(\mathbb{R})$ is called strong random Caratheodory if for each $\omega \in \Omega$,

(i) $(t,\omega) \mapsto F(t,x,\omega)$ is jointly measurable for each $x, y \in \mathbb{R}$, and

(ii) $x \to F(t,x,\omega)$ is Hausdorff continuous almost everywhere for $t \in J$.

Again, a strong random Caratheodory multi-valued function *F* is called strong L^1-Caratheodory if

(iii) For each real number *r* > 0 there exists a measurable function $h_r : \Omega \to L^1(J,\mathbb{R})$ such that for each $\omega \in \Omega$

$$\|F(t,x,\omega)\|_{\mathrm{P}} = \sup\{|u| : u \in F(t,x,\omega)\} \le h_r(t,\omega) \qquad \text{a.e.} \qquad t \in J$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

Then we have the following lemmas which are well-known in the literature.

Lemma 2.1 (Lasota and Opial) *Let E be a Banach space. If* $\dim(E) < \infty$ *and* $F : J \times E \times \Omega \to \mathrm{P}_{cp}(E)$ *is strong* L^1*-Caratheodory, then* $S^1_F(\omega)(x) \neq \emptyset$ *for each* $x \in E$.

Lemma2.2 (Caratheodory theorem) *Let E be a Banach space. If* $F : J \times E \to \mathrm{P}_{cp}(E)$ *is strong Caratheodory, then the multi-valued mapping* $(t,x) \mapsto F(t,x(t))$ *is jointly measurable for any measurable E-valued function x on J.*

3. Existence Result

Let X be a separable Banach space. A multi-valued mapping $Q : \Omega \to \mathrm{P}_p(X)$ is called measurable (respectively weakly measurable) if

$$Q^-(B) = \{\omega \in \Omega \,|\, Q(\omega) \cap B \neq \emptyset\} \in \mathrm{A} \tag{3.1}$$

for all closed (respectively open) subsets *B* in *X*. A multi-valued mapping $Q : \Omega \times X \to \mathrm{P}_p(X)$ is called a multi-valued random operator if $Q(\cdot, x)$ is measurable for each $x \in X$, and we write $Q(\omega, x) = Q(\omega)x$. A measurable function $\xi : \Omega \to X$ is called a random fixed point of the multi-valued random operator $Q(\omega)$ if $\xi(\omega) \in Q(\omega)\xi(\omega)$ for all $\omega \in \Omega$. The set of all random fixed

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

points of the multi-valued random operator $Q(\omega)$ is denoted by $\mathrm{F}_Q(\omega)$. A multi-valued random operator $Q:\Omega\times X\to \mathrm{P}_p(X)$ is called bounded (resp. totally bounded, compact, closed, completely continuous) if the multi-valued mapping $Q(\omega,\cdot)$ is bounded (resp. totally bounded, compact, closed, completely continuous) for each $\omega\in\Omega$. The details of compact and completely continuous operators appear in Granas and Dugundji [48].

Remark 3.1 If $Q_1,Q_2:\Omega\times X\to \mathrm{P}_p(X)$ are two multi-valued random operators, then the sum $Q_1+Q_2:\Omega\times X\to \mathrm{P}_p(X)$ defined by $\big(Q_1(\omega)+Q_2(\omega)\big)(x)=Q_1(\omega)x+Q_2(\omega)x$ is again a multi-valued random operator on *X*.

We employ the following random fixed-point theorem for completely continuous multi-valued mappings in Banach spaces.

Theorem3.1 (Dhage) *Let* (Ω,A) *be a measurable space, X a separable Banach space and let* $Q:\Omega\times X\to \mathrm{P}_{cp,cv}(X)$ *be continuous and condensing multi-valued random operator. Furthermore, if the set* $\xi=\{u\in M(\Omega,X)\,|\,\lambda(\omega)u\in Q(\omega)u\}$ *is bounded for all measurable functions* $\lambda:\Omega\to\mathbb{R}$ *with* $\lambda(\omega)>1$ *on* Ω*, then* $Q(\omega)$ *has a random fixed point, i.e., there is measurable function* $\xi:\Omega\to X$ *such that* $\xi(\omega)\in Q(\omega)\xi(\omega)$ *for all* $\omega\in\Omega$.

Remark 3.2 It is known that the compact and totally bounded multi-valued operators are condensing, but the converse may not be true.

We consider the following set of hypotheses in the sequel.

(A_1) $F(t,x,\omega)$ is compact-convex subset of $\mathbb{R}$ for all $(t,x,\omega)\in J\times\mathbb{R}\times\Omega$.

(A_2) *F* is strong random Caratheodory.

(A_3) There exists a measurable function $\gamma:\Omega\to L^1(J,\mathbb{R})$ with $\gamma(t,\omega)>0$ a.e. $t\in J$ and a continuous nondecreasing function $y:\mathbb{R}^+\to(0,\infty)$ such that for each $\omega\in\Omega$,

$$\|F(t,x,\omega)\|_{\mathrm{P}}\le\gamma(t,\omega)y\,(|x|).\quad a.e.\quad t\in J$$

for all $x\in\mathbb{R}$.

Theorem 3.2 *Assume that the hypotheses* $(A_1)-(A_3)$ *hold. Furthermore, if*

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

$$\int_{C}^{\infty} \frac{dr}{y(r)} > T \|\gamma(\omega)\|_{L^1} \tag{3.2}$$

for all $\omega \in \Omega$, *where* $C = |q_0(\omega)| + T|q_1(\omega)|$, *then the RDI (2.4.1) has a random solution in* $C(J, \mathbb{R})$ *defined on* $J \times \Omega$.

<u>Proof</u> Let $X = C(J, \mathbb{R})$. Define a multi-valued operator $Q : \Omega \times X \to \mathrm{P}_p(X)$ by

$$Q(\omega)x = \Big\{ u \in \mathrm{M}(\Omega, X) \,|\, u(t,\omega) = q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)v(s,\omega)ds, v \in S_F^1(\omega)(x) \Big\} \tag{3.3}$$

$$= \left(\mathrm{L} \circ S_F^1(\omega)\right)(x)$$

where $\mathrm{K} : \mathrm{M}\left(\Omega, L^1(J, \mathbb{R})\right) \to \mathrm{M}\left(\Omega, C^1(J, \mathbb{R})\right)$ is a continuous operator defined by

$$\mathrm{K}v(t,\omega) = q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)v(s,\omega)ds. \tag{3.4}$$

Clearly, the operator $Q(\omega)$ is well defined in view of hypothesis (H_2). We shall show that $Q(\omega)$ satisfies all the conditions of Theorem 3.1.

<u>Step</u> I : First, we show that Q is closed valued multi-valued random operator on $\Omega \times X$. Observe that the operator $Q(\omega)$ is equivalent to the composition $\mathrm{K} \circ S_F^1(\omega)$ of two operators on $L^1(J, \mathbb{R})$, where $\mathrm{L} : \mathrm{M}\left(\Omega, L^1(J, \mathbb{R})\right) \to X$ is the continuous operator defined by (3.4).

Next, we show that $Q(\omega)$ is a multi-valued random operator on *X*. First, we show that the multi-valued map $(\omega, x) \mapsto S_F^1(\omega)(x)$ is measurable. Let $f \in \mathrm{M}\left(\Omega, L^1(J, \mathbb{R})\right)$ be arbitrary. Then we have

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

$$
\begin{aligned}
d\left(f, S_F^1(\omega)(x)\right) &= \inf\left\{\left\| f(\omega)-h(\omega)\right\|_{L^1} : h\in S_F^1(\omega)(x)\right\} \\
&= \inf\left\{\int_0^T \left| f(t,\omega)-h(t,\omega)\right| dt : h\in S_F(\omega)(x)\right\} \\
&= \int_0^T \inf\left\{\left| f(t,\omega)-z\right| : z\in F\left(t,x(t,\omega),\omega\right)\right\} dt \\
&= \int_0^T d\left(f(t,\omega), F\left(t,x(t,\omega),\omega\right)\right) dt.
\end{aligned}
$$

But by hypothesis (A_2), the mapping $F\left(t,x\left(\eta(t),\omega\right),\omega\right)$ is measurable. Now the function $z\mapsto d\left(z,F(t,x,\omega)\right)$ is continuous and hence the mapping

$$
(t,x,\omega,f)\mapsto d\left(f(t,\omega), F\left(t,x\left(\eta(t),\omega\right),\omega\right)\right)
$$

is measurable from $J\times X\times\Omega\times L^1(J,\mathbb{R})$ into $\mathbb{R}^+$. Now the integral is the limit of the finite sum of measurable functions, and so, $d\left(f, S_F^1(\omega)(x)\right)$ is measurable. As a result, the multi-valued mapping $(\cdot,\cdot)\to S_{F(\cdot)}^1(\cdot)$ is jointly measurable.

Define the multi-valued map ϕ on $J\times X\times\Omega$ by

$$
\phi(t,x,\omega)=\left(\mathrm{K}\circ S_F^1(\omega)\right)(x)(t)=\int_0^t (t-s)F\left(s,x(s,\omega),\omega\right)ds.
$$

We shall show that $\phi(t,x,\omega)$ is continuous in *t* in the Hausdorff metric on $\mathbb{R}$. Let $\{t_n\}$ be a sequence in *J* converging to $t\in J$. Then we have

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

$$d_H\left(\phi(t_n,x,\omega),\phi(t,x,\omega)\right)$$

$$=d_H\left(\int_0^{t_n}(t_n-s)F\left(s,x(s,\omega),\omega\right)ds,\int_0^{t}(t-s)F\left(s,x(s,\omega),\omega\right)ds\right)$$

$$=d_H\left(\int_0^{t_n}(t_n-s)F\left(t,x(s,\omega),\omega\right)ds,\int_0^{t}(t_n-s)F\left(s,x(s,\omega),\omega\right)ds\right)$$

$$+d_H\left(\int_0^{t}(t_n-s)F\left(s,x(s,\omega),\omega\right)ds,\int_0^{t}(t_n-s)F\left(s,x(s,\omega),\omega\right)ds\right)$$

$$=d_H\left(\int_J \mathrm{X}_{[0,t_n]}(s)(t-s)F\left(t,x(s,\omega),\omega\right)ds,\right.$$

$$\left.\int_J \mathrm{X}_{[0,t]}(s)(t-s)F\left(s,x(s,\omega),\omega\right)ds\right)$$

$$+\int_0^{t}d_H\left((t_n-s)F\left(s,x(s,\omega),\omega\right),(t-s)F\left(s,x(s,\omega),\omega\right)\right)ds$$

$$=\int_J\left|\mathrm{X}_{[0,t_n]}(s)-\mathrm{X}_{[0,t]}(s)\right|\left|(t-s)\right|\left\|F\left(s,x(s,\omega),\omega\right)\right\|_{\mathrm{P}}\,ds$$

$$+\int_0^{t}\left|(t_n-s)-(t-s)\right|\left\|F\left(s,x(s,\omega),\omega\right)\right\|_{\mathrm{P}}\,ds$$

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

$$= \int_J \left| X_{[0,t_n]}(s) - X_{[0,t]}(s) \right| T \left\| F\left(s, x(s,\omega), \omega\right) \right\|_P \, ds$$

$$+ \int_0^T \left| (t_n - s) - (t - s) \right| \left\| F\left(s, x(s,\omega), \omega\right) \right\|_P \, ds$$

$$= \int_J \left| X_{[0,t_n]}(s) - X_{[0,t]}(s) \right| \gamma(s,\omega) y \left(\left| x(s,\omega) \right| \right) ds$$

$$+ \int_0^T \left| t_n - t \right| \left\| F\left(s, x(s,\omega), \omega\right) \right\|_P \, ds$$

$$= \int_J \left| X_{[0,t_n]}(s) - X_{[0,t]}(s) \right| \gamma(s,\omega) y \left(\left\| x(\omega) \right\| \right) ds$$

$$+ \int_0^T \left| t_n - t \right| \left\| F\left(s, x(s,\omega), \omega\right) \right\|_P \, ds$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

Thus the multi-valued map $t \mapsto \phi(t, x, \omega)$ is continuous and hence, by Lemma 3.2, the map $(t, x, \omega) \mapsto \int_0^t (t - s) F\left(s, x(s,\omega), \omega\right) ds$ is measurable. Again, since the sum of two measurable multi-valued functions is measurable, the map

$$(t, x, \omega) \mapsto q_0(\omega) + q_2(\omega) t + \int_0^t (t - s) F\left(s, x(s,\omega), \omega\right) ds$$

is measurable. Consequently, $Q(\omega)$ is a random multi-valued operator on [*a*, *b*].

<u>Step</u> II : Next, we show that $Q(\omega)$ is totally bounded and continuous on bounded subsets of *X* for each $\omega \in \Omega$. Let *S* be a bounded subset of *X*. Then there is real number *r* > 0 such that $\|x\| \le r$ for all $x \in S$. First, we show that $Q(\omega)$ is a continuous multi-valued random operator on *X*. Let $\{x_n\}$ be a sequence in *S* converging to a point *x*. Then by Hausdorff continuity of the multi-valued mapping $F(t, x, \omega)$ in *x* and by the dominated convergence theorem, we obtain

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

$$\lim_{n\to\infty} Q(\omega)x_n(t) = q_0(\omega) + q_2(\omega)t + \lim_{n\to\infty}\int_0^t (t-s)F\left(s, x_n(s,\omega),\omega\right)ds$$

$$= q_0(\omega) + q_2(\omega)t + \int_0^t \lim_{n\to\infty}(t-s)F\left(s, x_n(s,\omega),\omega\right)ds$$

$$= q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)F\left(s, x(s,\omega),\omega\right)ds$$

$$= Q(\omega)x(t)$$

for all $t \in J$ and $\omega \in \Omega$. This shows that $Q(\omega)$ is a Hausdorff continuous multi-valued random operator on *X*.

Next we show that $Q(\omega)$ is totally bounded operator on *X* for each $\omega \in \Omega$. Let $\{y_n(\omega)\}$ be a sequence in $\bigcup Q(\omega)(S)$ for some $\omega \in \Omega$. We will show that $\{y_n(\omega)\}$ has a cluster point. This is achieved by showing that $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous sequence in *X*.

<u>Case</u> I : First, we show that $\{y_n(\omega)\}$ is uniformly bounded sequence. By the definition of $\{y_n(\omega)\}$, we have a $v_n(\omega) \in S_F^1(\omega)(x_n)$ for some $x_n \in S$ such that

$$y_n(t,\omega) = q_0(\omega) + q_1(\omega)t + \int_0^t (t-s)v_n(s,\omega)ds, \quad t \in J .$$

Therefore,

$$\left|y_n(t,\omega)\right| \le \left|q_0(\omega)\right| + T\left|q_1(\omega)\right| + \int_0^t \left|(t-s)\right|\left|v_n(s,\omega)\right|ds$$

$$\le \left|q_0(\omega)\right| + T\left|q_1(\omega)\right| + \int_0^t \left|(t-s)\right|\left\|F\left(s, x_n(s,\omega),\omega\right)\right\|_{\mathrm{P}}\, ds$$

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

$$\leq |q_0(\omega)| + T|q_1(\omega)| + \int_0^t T\gamma(s,\omega)y\left(\|x_n(\omega)\|\right)$$

$$\leq |q_0(\omega)| + T|q_1(\omega)| + T\|\gamma(\omega)\|_{L^1}y(r)$$

for all $t \in J$. Taking the supremum over t in the above inequality yields,

$$\|y_n(\omega)\| \leq |q_0(\omega)| + T|q_1(\omega)| + T\|\gamma(\omega)\|_{L^1}y(r)$$

which shows that $\{y_n(\omega)\}$ is a uniformly bounded sequence in $Q(\omega)(X)$.

Next we show that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)(X)$. Let $t,\tau \in J$. Then, for each $\omega \in \Omega$, we have

$$|y_n(t,\omega) - y_n(\tau,\omega)| = \left|\int_0^t (t-s)v_n(s,\omega)ds - \int_0^\tau k(\tau,s,\omega)v_n(s,\omega)ds\right|$$

$$\leq \left|\int_0^t (t-s)v_n(s,\omega)ds - \int_0^t (\tau-s)v_n(s,\omega)ds\right|$$

$$+\left|\int_0^t (\tau-s)v_n(s,\omega)ds - \int_0^\tau (\tau-s)v_n(s,\omega)ds\right|$$

$$\leq \left|\int_0^t |(t-s)-(\tau-s)||v_n(s,\omega)|ds\right|$$

$$+\left|\int_\tau^t |(\tau-s)||v_n(s,\omega)|ds\right|$$

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Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

$$\leq \int_0^T |(t-s)-(\tau-s)| \left\| F\left(s, x_n(s,\omega),\omega\right)\right\|_{\mathrm{P}} \, ds$$

$$+ \left| \int_\tau^t |\tau - s| \left\| F\left(s, x_n(s,\omega),\omega\right)\right\|_{\mathrm{P}} \, ds \right|$$

$$\leq \int_0^T |t-\tau| \gamma(s,\omega) y\left(\|x(\omega)\|\right) ds$$

$$+ \left| \int_\tau^t T\gamma(s,\omega) y\left(\|x(\omega)\|\right) ds \right|$$

$$\leq \int_0^T |t-\tau| \gamma(s,\omega) y\,(r) ds$$

$$+ \left| p(t,\omega) - p(\tau,\omega) \right|,$$

where, $p(t,\omega) = \int_0^t T\gamma(s,\omega) y\,(r) ds.$ From the above inequality, it follows that

$$|y_n(t,\omega) - y_n(\tau,\omega)| \to 0 \text{ as } t \to \tau .$$

This shows that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)(X).$ Now $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous for each $\omega \in \Omega$, so it has a cluster point in view of Arzela-Ascoli theorem. As a result, $Q(\omega)$ is a compact multi-valued random operator on *X*. Thus $Q(\omega)$ is a continuous and totally bounded and hence completely continuous multi-valued random operator on *X*.

<u>Step</u> III : Next, we show that $Q(\omega)$ has convex values on *X* for each $\omega \in \Omega$. Again, let $u_1, u_2 \in Q(\omega)x.$ Then there are $v_1, v_2 \in S_F^1(\omega)(x)$ such that

$$u_1(t) = q_0(\omega) + q_2(\omega)t + \int_0^t (t-s) v_1(s,\omega) ds, \quad t \in J,$$

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

and $$u_2(t) = q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)v_2(s,\omega)ds, \quad t \in J.$$

Now for any $\lambda \in [0,1]$,

$$\lambda u_1(t,\omega) + (1-\lambda)u_2(t,\omega) = \lambda\left(q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)v_1(s,\omega)ds \right)$$

$$+(1-\lambda)\left(q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)v_2(s,\omega)ds \right)$$

$$= q_0(\omega) + q_2(\omega)t + \int_0^t (t-s)\left[v_1(s,\omega) + (1-\lambda)v_2(s,\omega)\right]ds.$$

Since $S_F^1(\omega)$ has convex values on X (because F has convex values), we have that $v(t,\omega) = \lambda v_1(t,\omega) + (1-\lambda)v_2(t,\omega) \in S_F^1(\omega)(x)(t)$ for all $t \in J$. Hence, $\lambda u_1 + (1-\lambda)u_2 \in Q(\omega)x$ and consequently $Q(\omega)x$ is convex for each $x \in X$. As a result, $Q(\omega)$ defines a multi-valued random operator $Q : \Omega \times X \to \mathrm{P}_{cp,cv}(X)$.

<u>Step</u> IV : Finally, we show that the set ξ is bounded. Let $u \in \mathrm{M}\left(\Omega, C(J,\mathbb{R})\right)$ such that $\lambda u(t,\omega) \in Q(\omega)u(t)$ on $J \times \Omega$ for all $\lambda > 1$. Then there is a $v \in S_F^1(\omega)(u)$ such that

$$u(t,\omega) = \lambda^{-1}q_0(\omega) + q_2(\omega)t + \lambda^{-1}\int_0^t (t-s)v(s,\omega)ds$$

for all $t \in J$ and $\omega \in \Omega$. Therefore,

$$|u(t,\omega)| \le |q_0(\omega) + q_2(\omega)t| + \int_0^t |(t-s)||v(s,\omega)|ds$$

$$\le |q_0(\omega)| + T|q_1(\omega)| + \int_0^t T\left\|F\left(s,u(s,\omega)\right)\right\|_{\mathrm{P}} ds$$

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

$$\leq |q_0(\omega)| + T|q_1(\omega)| + \int_0^t T\gamma(s,\omega)y\left(|u(s,\omega)|\right)ds$$

for all $t \in J$ and $\omega \in \Omega$.

Let $m(t,\omega) = \sup_{s\in[0,t]} |u(s,\omega)|$. Then, we have $|u(t,\omega)| \leq m(t,\omega)$ for all $(t,\omega) \in J \times \Omega$. Furthermore, there is a point $t^* \in [0,t]$ such that $m(t,\omega) = |u(t^*,\omega)|$. Hence, we have

$$\begin{aligned} m(t,\omega) &= |u(t^*,\omega)| \\ &\leq |q_0(\omega)| + T|q_1(\omega)| + \int_0^{t^*} T\gamma(s,\omega)y\left(|u(s,\omega)|\right)ds \\ &\leq C + \int_0^t T\gamma(s,\omega)y\left(m(s,\omega)\right)ds \end{aligned}$$

where $C = |q_0(\omega)| + T|q_1(\omega)|$. Put

$$w(t,\omega) = C + \int_0^t T\gamma(s,\omega)y\left(m(s,\omega)\right)ds.$$

Differentiating w.r.t. t,

$$\left.\begin{aligned} w'(t,\omega) &= T\gamma(t,\omega)y\left(m(t,\omega)\right) \\ w(0,\omega) &= C \end{aligned}\right\} \quad (3.5)$$

for all $t \in J$ and $\omega \in \Omega$.

From the above expression, we obtain

$$\left.\begin{aligned} \frac{w'(t,\omega)}{y\left(w(t,\omega)\right)} &\leq T\gamma(t,\omega) \\ w(0,\omega) &= C \end{aligned}\right\}. \quad (3.6)$$

Variorum Multi- Disciplinary e-Research Journal
Vol.-01, Issue-III, February 2011

Integrating the above inequality from 0 to *t*,

$$\int_0^t \frac{w'(s,\omega)}{y\left(w(s,\omega)\right)} ds \le \int_0^t T\gamma(s,\omega) ds.$$

By change of the variables,

$$\int_C^{w(t,\omega)} \frac{dr}{y(r)} \le T\left\|\gamma(\omega)\right\|_{L^1} < \int_C^{\infty} \frac{dr}{y(r)}.$$

Now an application of the mean value theorem yields that there is a constant $M(\omega) > 0$ such that

$$\left|u(t,\omega)\right| \le m(t,\omega) \le w(t,\omega) \le M(\omega)$$

for all $t \in J$ and $\omega \in \Omega$. Hence by Theorem 3.1, the RII, has a random solution on $J \times \Omega$. This completes the proof. □

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